CONICAL BODIES OF MINIMUM DRAG IN HYPERSONIC GAS FLOW

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In [1] an optimum shape of a three-dimensional slender body was found on the basis of Newtonian flow. The restriction for thickness of the body was dictated by the fundamental necessity to reduce the problem to solving an ordinary differential equation, since otherwise, the extremals are determined by a complicated nonlinear second order partial differential equation. In some cases, however, retaining the expression for the drag obtained in [1], and restricting the class of admissible surfaces, it is possible to find without significant complications the optimum shapes of a thick three-dimensional body. For example, in the class of arbitrary conical surfaces $(f(x) = x,\ 0 \leqslant x \leqslant 1$, Fig.1) the cross section of the body of minimum drag is determined by the condition of the minimum of the functional

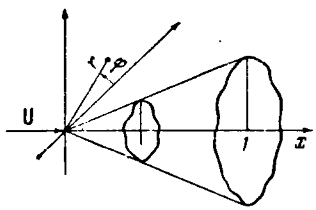

Fig. 1

$$C_x = \frac{1}{S}\int_0^{2\pi} \frac{r^4(\varphi)\,d\varphi}{1 + r^2(\varphi) + r'^2(\varphi)\,/\,r^2(\varphi)} \qquad \left(S = \frac{1}{2}\int_0^{2\pi} r^2\,d\varphi\right) \qquad (0.1)$$

Here the drag coefficient C_x (C_D) is associated with the maximum cross-section area S. We note in passing that a drag formula similar to (0.1) has been applied in his studies by G.I.Maikapar.

Below, we consider only conical bodies.

1. The corresponding variational problem is formulated in the following fashion. In the class of smooth curves, possessing a finite number of points of discontinuity in the first derivative, we are to determine a closed curve, for which the functional (0.1) assumes a minimum. As additional conditions, we shall consider that the maximum cross-section area S and a characteristic radial dimension r_0 are given. (All quantities are referred to the body length, assumed equal to unity). Similar variational problems, as is well known, are reduced to the solution of Euler's equation for the function $F = \lambda r^2 + r^4 (1 + r^2 + r'^2 r^{-2})^{-1}$. Moreover, along the extremals, the necessary Legendre condition $F_{r'r'} \geqslant 0$, must be satisfied, and at the points of discontinuity of the derivatives, the Weierstrass-Erdman condition

$$\alpha_{\varphi-} = \alpha_{\varphi+}, \qquad \alpha = \frac{1 + r^2 + 3r'^2 r^{-2}}{(1 + r^2 + r'^2 r^{-2})^2}, \qquad \beta_{\varphi-} = \beta_{\varphi+}, \qquad \beta = \frac{r' r^{-1}}{(1 + r^2 + r'^2 r^{-2})^2}$$

must be satisfied.

The function F contains no independent variables, so that the corresponding Euler equation admits the integral

$$\frac{r^4}{(1 + r^2 + r'^2 r^{-2})} + \frac{2r^2 r'^2}{(1 + r^2 + r'^2 r^{-2})^2} + \lambda r^2 = C \tag{1.2}$$

In particular, it follows from (1.2) that $C - \lambda r^2 \geqslant 0$. Let us introduce the new variables p and z. In terms of these variables equation (3) is written in the form

$$p^2 = \frac{3 \pm \sqrt{9 - 8z^2}}{2z^2} - 1, \quad p^2 = \frac{r'^2}{r^2 (1 + r^2)}, \quad z^2 = \frac{(C - \lambda r^2)(1 + r^2)}{r^4} \tag{1.3}$$

We consider the sign of the square root. From the condition that the quantity under the square root sign be non-negative, we see that $z^2 \leqslant {}^9/_8$, from which we get $p^2 \leqslant {}^1/_3$. for the negative sign of the root. On the other hand, according to Legendre's condition, the lower bound of p^2 equals $^1/_3$. Consequently, the sign in front of the root could only be positive; and, as z^2 decreases from $^9/_8$ to 0, p^2 increases from $^1/_3$ to ∞.

We note one property essential for later discussion. From the conditions on the discontinuities (1.1), which may be expressed entirely in terms of p, it follows that crossing from one integral curve to another can occur only either $p_{\varphi\pm} = \pm\infty$, or at the points of fixed radius $r = r_0$. This shows that in order to construct closed extremal curves from integral curves of different families, it is necessary that the variable z assume the value zero. Then from (1.3), if $\lambda > 0$, we shall have $r^2 \leqslant C/\lambda$, and $C > 0$, and the solution corresponding to this inequality may be constructed if the characteristic dimension r_0 is the minimum value of the radius.

If we assume that $\lambda < 0$, then we correspondingly obtain $r^2 \geqslant C/\lambda$ and $C < 0$. This case corresponds to the problem in which the maximum radius is the characteristic dimension. The two cases can be studied in an entirely similar manner, so that in what follows we shall consider only the first case. We integrate Equation (1.3) from r_0 to r, and setting $r^2 = r_0{}^2 t$, we get

$$\varphi + C_1 = \pm \frac{1}{2} \int_1^t \frac{q(z)\, dt}{t\sqrt{1 + r_0{}^2 t}} \; (1 \leqslant t \leqslant t_0), \qquad t_0 = \frac{C}{\lambda r_0{}^2} \tag{1.4}$$

$$q(z) = z\sqrt{2}\,(3 - 2z^2 + \sqrt{9 - 8z^2})^{-1/2}, \qquad z^2 = \lambda (t_0 - t)(t + 1/r_0{}^2)\, t^{-2}$$

We carry out the determination of the constants t_0 and λ. From the condition of the closed extremal and the isoperimetric condition, we have

$$\int_0^{t_0} \frac{q(z)\, dt}{t\sqrt{1 + r_0{}^2 t}} = \frac{2\pi}{n}, \qquad \int_0^{t_0} \frac{q(z)\, dt}{\sqrt{1 + r_0{}^2 t}} = \frac{2\pi s}{n}, \qquad s = \frac{S}{\pi r_0{}^2} \tag{1.5}$$

Here n is the number of double sections from which the extremal is composed. Formula (1.5), starting with some number n, permits the determination of the constants C and λ for each given value of this quantity. As a result, we obtain a countable set of extremals, satisfying all the conditions of the variational problem. The form of the cross-sections is completely determined by the parameters r_0 and s. Let us now calculate the drag of the optimum body thus determined. According to (0.1), the drag coefficient may be written in the form

$$C_x = \frac{2\sqrt{2}\, n r_0 \lambda^{3/2}}{S} \int_0^{t_0} \frac{(t_0 - t)^{3/2}\, dt}{t^2 (3 + \sqrt{9 - 8z^2})(3 - 2z^2 + \sqrt{9 - 8z^2})^{1/2}} \tag{1.6}$$

2. The relations thus obtained are complicated for actual computations, so that we shall introduce below some simple approximate formulas for calculating solutions. Consider the class of aptimum bodies, for which the parameter $\lambda(t_0 - 1)(1 + r_0^{-2}) \ll 1$. Then, according to (1.4), $z^2 \ll 1$. Rejecting in the expression for $q(z)$ terms of order $O(z^3)$, we obtain from (1.4) the equation of the cross-section contour in the form

$$\varphi = \pm \frac{\lambda^*}{n}\left[\sqrt{t_0 - 1} - \frac{\sqrt{t_0 - t}}{t} + \frac{1}{\sqrt{t_0}} \ln \frac{(\sqrt{t_0} - \sqrt{t_0 - 1})(\sqrt{t_0} + \sqrt{t_0 - t})}{\sqrt{t}}\right] + C_1$$

$$\lambda^* = \frac{n}{2r_0}\left(\frac{\lambda}{3}\right)^{1/2} \qquad (2.1)$$

Using (1.5), we see that the constants t_0 and λ^* are determined from the relations

$$\lambda^* \left[\sqrt{t_0 - 1} - \frac{1}{\sqrt{t_0}} \ln(\sqrt{t_0} + \sqrt{t_0 - 1})\right] = \pi$$

$$\lambda^* \left[\sqrt{t_0} \ln(\sqrt{t_0} + \sqrt{t_0 - 1}) - \sqrt{t_0 - 1}\right] = {}^1/_2 \pi s \qquad (2.2)$$

According to the latter equations, the constants t_0 and λ^* depend only on the parameter s. The drag coefficient (1.6) in this case is given by Expression

$$C_x = \frac{8\lambda^{*3} r_0^4}{S n^2}\left[(t_0 - 1)^{3/2} + 3(t_0 - 1)^{1/2} - 3\sqrt{t_0}\ln(\sqrt{t_0} + \sqrt{t_0 - 1})\right] \qquad (2.3)$$

This asymptotic formula is free from any restriction on the thickness of the body. As is clear from the last relations, for any thickness of the body, the shape of the cross-section does not depend on r_0. For slender bodies, this fact always occurs [1]. Consequently, even though in Expression (1.4) the quantity r_0 is present, we may expect that in general the dependence of the shape on this parameter will be weak. However, this approximation understates significantly the function $q(z)$, and as a result the approximate formulas are only valid for large values of n and may not be

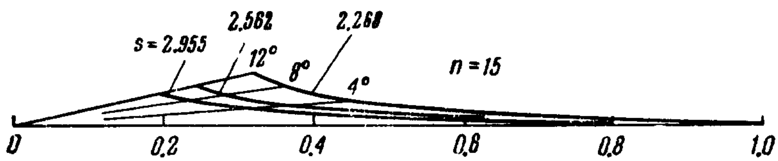

Fig. 2

used to estimate the smallest possible values of this number. In Fig. 2 there are shown segments of the contour, from which the cross-section of a star-shaped body may be constructed. A calculation was carried out using (2.1) for the number of points $n = 15$ for several values of the parameter s. Comparison with the results obtained from the slender body formulas [1] shows that for large values of s the contours coincide. Consequently, the gain in the drag will be of the same order as in the slender body case. In fact, we calculate the value of the relative drag $L = n^2 C_x / C_x^\circ$ (C_x° = drag coefficient (C_D) of an equivalent circular cone) depending on the parameter s and the thickness parameter r_0. The calculated values of L as a function of s for different values of r_0 according to Formula (2.3) are shown in Fig. 3. From the curves, it is clear that for increased body thickness, the relative drag increases; the behavior of the drag is analogous to the case of the slender body, and for $n = 15$, it is smaller than the drag of an equivalent circular cone by roughly 20 times. In Fig. 3, the corresponding values of the drag computed by the slender body formula (in formula (17) of [1] the factor 0.25 is missing) agree with the drag curve for thick cone with $r_0 = 0$.

The results obtained above are based on the pressures determined by Newton's formula. For flows around simple bodies (bodies of revolution, cones at angles of attack, etc.), the Newtonian formula gives good approximations, as is shown by comparison with experiments, starting from Mach 5 and higher.

For bodies of more complicated configurations, there are no reliable experimental results in the literature to permit us to judge the validity of the Newtonian theory. Thus, as a useful error estimate, we have the following statement. For hypersonic flow past pointed bodies ($M \to \infty$), there occurs an attached shock wave, which differs from the body surface by an angle of the order

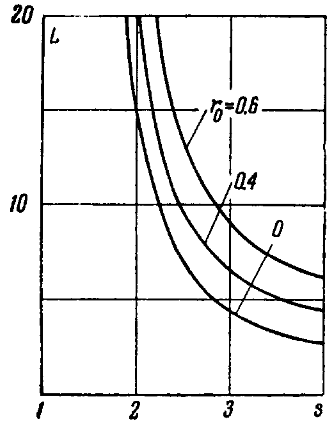

Fig. 3

$$O(\varepsilon)\ (\varepsilon = (\varkappa - 1)(\varkappa + 1)^{-1}$$

$\varkappa$ is the ratio of specific heats. Value of this angle determines the correction for the Newtonian formula for slightly curved surfaces.

In the given case, the surface of the optimum body obtained is concave (if we set $r_0 = r_{max}$, then the surface will be convex). However, according to Fig.2, the curvature of the cross-section contours is small, except for a region of small radii in the neighborhood of the stagnation point. Consequently, the pressure by the Newtonian formula will be close to the true value over a large part of the body surface. As to the region in the neighborhood of the edge, there may actually occur a significant deflection towards the rise of pressure. This means that the predicted decrease in wave drag by a factor of twenty is an exaggeration. It is also clear that for large values of n, the boundary layer completely fills the space between the points, and the Newtonian flow pattern will not occur.

3. The solution of the variational problems for slender bodies and for the case considered permits the construction of optimum cross-section contours only for those values of the parameter s which are included between the defined limits.

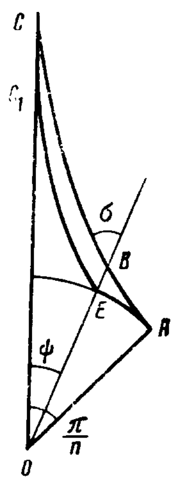

Fig. 4

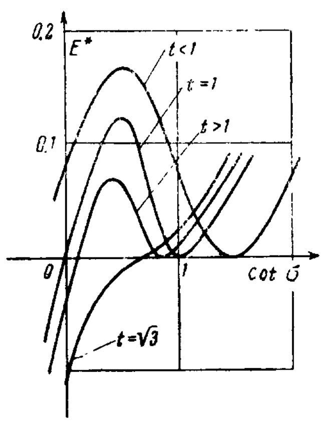

Fig. 5

In particular, for a fixed but not very large value of n, the solution does not exist when the parameter s is near to unity. Nevertheless, it is of interest to find the shape of the cross-section contour for a minimum drag body with a fixed number of points n as s tends to 1. To this end, let us consider smooth admissible curves with a finite number of discontinuities of the derivatives, at which the angle σ between the polar radius and the tangent satisfies the inequality

$$0 \leqslant \sigma \leqslant \pi/2.$$

The latter is necessary to exclude those admissible curves which are wavy-shaped and close to extremals with n larger than that given. Let the extremal ABC (Fig.4) correspond to the minimum value of the parameter s, such that for smaller value of the parameter, there are no two-sided extremals. Then the smallest value of the drag may be attained at the boundary with $\sigma = 0$ or $\sigma = \pi/2$. The first possibility $\sigma = 0$ can never be realized, so that we shall consider the case $\sigma = \pi/2$, which corresponds to an optimum contour consisting of the arc AE of the circle and the extremal EC_1. Using the general formula of the first variation, one easily establishes the fact that for a slender body, the angle σ at the point E is

$\sigma_E = \pi / 4$. The parameters of the contour and the location of the point E (angle ψ) are determined with the aid of the formulas in [1].

It now remains to clarify, what is to be done in the case when the parameter s decreases to such a value that the extremal at point A has the angle $\sigma = \pi/4$, and moreover, there exist extremals of the type ABC, corresponding to still smaller values of this parameter; i.e. there exists a range of values of the parameter s, for which it is possible to construct two optimum contours: one of the type ABC, the other of type AEC_1. Depending on the form of the functional, either of the contours may correspond to the least drag. The investigation may be carried through in the following manner. Consider the necessary Weierstrass tast, for which along the extremal $E(\varphi, r, r', R) \geqslant 0$ for all possible admissible elements (φ, r, R). After some simple calculations, this condition may be written as

$$E^* = \frac{E\,(1+t^2)}{r^4 t^4 \sin^2\sigma} = \left(\cot^3\sigma - \frac{3+t^2}{2t}\cot^2\sigma + \cot\sigma + \frac{1-t^2}{2t^3}\right) \geqslant 0 \qquad \left(t = \frac{r}{r'}\right)$$

The sign of the function E is determined by the sign of the expression in parentheses. Results of a qualitative study shown in Fig.5 indicate that if along the extremal $0 \leqslant t \leqslant 1$, everywhere, then $E^* \geqslant 0$ for any positive angles σ. If however an extremal contains a piece along which $1 \leqslant t \leqslant \sqrt{3}$, then on that extremal the minimum is not attained. This leads to the conclusion that for values of the parameter $t_0 > 1$ it is necessary to pass to boundary extremum. The drag of the optimum body, as the parameter s tends to unity, will rise sharply to the value corresponding to the drag of a circular cone. Investigation of boundary extrema for conical bodies offers no difficulties, but the angle σ_E is somewhat different from that given by Formula $\sigma_E = \tan^{-1}\sqrt{1 + r_0^2}$.

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